Dust acoustic waves in non-equilibrium compressed dusty plasmas

DAW: the moment approach

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Пылеакустические волны в неравновесной сжатой пылевой плазме

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- We consider a model OCP consisting of dust particles of invariable charge and size and with the effective interaction screened by other subsystems; the charges, masses, number densities and temperature of each species are: $Z_a e$, m_a, n_a, and T_a $(a = e, i, d)$. The dimensionless parameter

$$\begin{aligned} \Gamma &= \beta_d \left(Z_d e\right)^2 / a_d \\ \beta_d^{-1} &= k_B T_a \quad , \; a_d = \sqrt[3]{3/4\pi n_d} \end{aligned}$$

can be $\Gamma \gtrsim 1$ under normal conditions: the traditional approaches are inapplicable and the interaction effects determine the physical properties of dusty plasmas, in particular, the dust acoustic waves (DAW) dispersion.

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- We wish to outline a mathematical approach which can be employed in this context and in other fields of research.
- We will compare the "dynamic" results obtained within this moment approach (based on the sum-rules and other exact relations) to those of the HD model, Pis'ma Zh. Eksp. Teor. Fiz., **91** (2010) 626-633.

Problem

Let $f(t)$ be a positively definite integrable function (distribution density), e.g., given only by a set of its numerical values.
Find an analytic representation of $f(t)$ such that $f(t)$ were an imaginary part of the Nevanlinna class (response) function

$$F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt, \ \operatorname{Im} z > 0 \tag{1}$$

analytic in $\operatorname{Im} z > 0$, continuous on the real axis $\operatorname{Im} z = 0$ and such that

$$f(t) = \operatorname{Im}\left(\lim_{\eta \downarrow 0} F(t + i\eta)\right) > 0.$$

Fact

Assume that we are able to compute some power moments of the function in question,

$$c_k = \frac{1}{\pi}\int_{-\infty}^{\infty} t^k f\left(t\right) dt \ , \qquad k = 0, 1, \ldots, 2n, \tag{2}$$

then, asymptotically, along any ray in $\mathrm{Im}\, z \geq 0$*,*

$$F(z \to \infty) = -\frac{1}{\pi z}\int_{-\infty}^{\infty} \frac{f(t)}{1 - t/z} dt \simeq$$

$$\simeq -\frac{c_0}{z} - \frac{c_1}{z^2} - \frac{c_2}{z^3} - \cdots - \frac{c_{2n}}{z^{2n+1}} + \mathrm{o}\left(\frac{1}{z^{2n+1}}\right). \tag{3}$$

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- The non-canonical solutions are continuous and are described by Nevanlinna's formula,

$$\int_{-\infty}^{\infty} \frac{f(t)dt}{z-t} = \frac{E_{n+1}(z) + Q_n(z)\, E_n(z)}{D_{n+1}(z) + Q_n(z)\, D_n(z)} , \qquad (4)$$

where the polynomials $\{D_\nu(z)\}$ can be obtained from the basis $\{1, z, z^2, \ldots\}$ by the Gramm-Schmidt procedure with the weight $f(t)$, and $\{E_\nu(z)\}$ are the conjugate polynomials; the parameter function $Q_n(z)$ also belongs to the class of response functions and is such that $\lim_{z\to\infty} Q_n(z)/z = 0$, $\mathrm{Im}\, z > 0$.

Fact

We have applied, and quite successfully, the moment approach to a number of equilibrium strongly coupled Coulomb systems, see some of our latest publications and references therein:

1. *Phys. Rev. E, 81 (2010) 026402;*
2. *Phys. Rev. Lett., 101 (2008) 075002;*
3. *Phys. Rev. E, 76 (2007) 026403;*
4. *Phys. Rev. B, 75 (2007) 115109;*
5. *Contrib. Plasma Phys. 43 (2003) 252- 257.*

- The starting point in the application of the method of moments [2] to **dusty plasmas** is the assumption on the existence of the system inverse dielectric function, the genuine response function of the system we consider, $\epsilon^{-1}(k, \omega)$. This method takes into account the (convergent) sum rules and other exact relations, like the compressibility sum rule, automatically.

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- The sum rules we employ are actually the (positive) power frequency moments of the loss function

$$\mathcal{L}(k,\omega) = -\frac{\operatorname{Im}\epsilon^{-1}(k,\omega)}{\omega} : \tag{5}$$

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- Notice that the odd-order moments vanish due to the symmetry of the loss function.
- Let us introduce also the frequencies $\omega_1 = \omega_1(k)$ and $\omega_2 = \omega_2(k)$:

$$\omega_1^2 = \omega_1^2(k) = C_2/C_0(k), \quad \omega_2^2 = \omega_2^2(k) = C_4(k)/C_2 . \tag{7}$$

The interaction potential between the dust particles screened by the environment in the neutral OCP model of dusty plasmas is usually approximated by the Yukawa potential,

$$\varphi_Y(r) = \frac{Z_d^2 e^2}{r} \exp\left(-\frac{r}{\lambda}\right), \tag{8}$$

- We employ the two-exponent model potential [3],

$$\varphi(r) = \frac{Z_d^2 e^2}{r} \left(\theta_1 \exp(-k_1 r) + \theta_2 \exp(-k_2 r)\right) \quad , \qquad (9)$$

which takes into account the external sources, the sinks, the recombination, etc. The temperature is that of the dust subsystem, while other species', the environment, temperatures, might be different.

- Here

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2. $k_2^{-1} = \sqrt{\frac{D_i(1+\beta_i/\beta_e)}{2\beta_{ei} n_0}}$ is the average travel of an ion in the process of ambipolar diffusion during the recombination time, D_i is the ionic diffusion coefficient, β_{ei} is the electron–ion recombination coefficient and n_0 is the concentration of ions or electrons in the non-perturbed plasma.

One can use this effective potential to construct the interaction contribution into the dust OCP Hamiltonian in the second-quantization picture:

$$\hat{H} = \hat{K} + \hat{H}_{int} = \tag{10}$$

$$= \frac{\hbar^2}{2m_d} \sum_{\mathbf{q}} q^2 a_{\mathbf{q}}^{+} a_{\mathbf{q}} + \sum_{\mathbf{q}} \varphi(q) \left(n_{\mathbf{q}} n_{-\mathbf{q}} - n_{\mathbf{0}} \right) \quad , \tag{11}$$

where

$$\varphi(k) = 4\pi e^2 Z_d^2 \left(\frac{\theta_1}{k^2 + k_1^2} + \frac{\theta_2}{k^2 + k_2^2} \right) = \phi(k)\, \zeta(k)\,, \tag{12}$$

$$\phi(k) = \frac{4\pi e^2 Z_d^2}{k^2}.$$

The convergence of the moment $C_0(k)$ follows, by virtue of the Kramers-Kronig dispersion relations, from the existence of the static inverse dielectric function,

$$\epsilon^{-1}(k,0) = \lim_{\eta\downarrow 0} \epsilon^{-1}(k, i\eta) = 1 + \int_{-\infty}^{\infty} \mathrm{Im}\epsilon^{-1}(k,\omega) \frac{d\omega}{\pi\omega} = \tag{13}$$

$$= 1 + \int_{0}^{\infty} \mathrm{Im}\epsilon^{-1}(k,\omega) \frac{d(\omega^2)}{\pi\omega^2} = 1 - C_0(k). \tag{14}$$

Additionally, if we refer to the classical version of the fluctuation-dissipation theorem:

$$\mathcal{L}\left(k,\omega\right) = \pi\beta_{d}\phi\left(k\right)S\left(k,\omega\right), \tag{15}$$

where $S\left(k,\omega\right)$ is the corresponding dust dynamic structure factor, then,

$$C_0\left(k\right) = \frac{k_g^2}{k^2}S\left(k\right), \quad k_g = \sqrt{4\pi e^2\beta_{d}Z_d^2 n_d}\ , \tag{16}$$

and the dust static structure factor can be calculated independently, e.g., in the HNC or MD approximations.

- For the power frequency moments (6) directly from the Kubo linear reaction theory formula we get the f-sum rule:

$$C_2(k) = \frac{n_d k^2}{m_d}\phi(k) = \frac{4\pi Z_d^2 e^2 n_d}{m_d} = \omega_{pd}^2 \ . \tag{17}$$

And neglecting quantum corrections,

$$C_4(k) = \omega_{pd}^4 \left(\zeta(k) + V(k) + U(k)\right), \tag{18}$$

where

$$V(k) = 3k^2/k_g^2 = k^2 c_s^2/\omega_{pd}^2, \tag{19}$$

$$U(k) = \frac{1}{2\pi^2 n_d}\int_0^\infty p^2\left(S(p) - 1\right)\sum_{i=1,2} f_i(p,k)\,dp, \tag{20}$$

coincides with the QLCA model "dynamic matrix" of [4] for the Yukawa model.

Here, the angular factors are

$$f_i\left(p,k\right)=\frac{3}{4}-\frac{p^2+k_i^2}{4k^2}-\frac{1}{3}\frac{p^2}{p^2+k_i^2}+$$
$$+\frac{\left(p^2+k_i^2-k^2\right)^2}{16pk^3}\ln\frac{k_i^2+\left(p+k\right)^2}{k_i^2+\left(p-k\right)^2}\quad .$$

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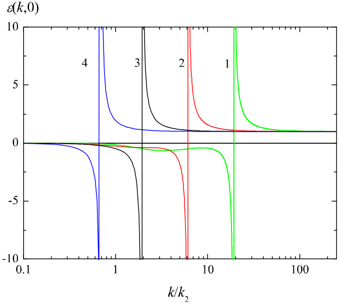
- We have computed the moments $C_0\left(k\right)$ and $C_4\left(k\right)$ for the potential $\varphi\left(k\right)$.
- The dust radial distribution function (RDF) and the static structure factor, $S\left(k\right)$, were calculated by an adequate solution of the Ornstein-Zernike equation in the HNC approximation and by the method of molecular dynamics (MD).

We have also estimated the static dielectric function from the compressibility sum rule (14) and the FDT (16):

$$\epsilon(k,0) = \frac{k^2}{k^2 - k_g^2 S(k)}. \quad (21)$$

We present, as an example, the results for different values of the pressure and for $n_d = 10^5\ cm^{-3}$, $k_g = 511.38\ cm^{-3}$:



Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}(k,\omega)}{z-\omega}d\omega =$$

$$= -\frac{1}{\pi z}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\epsilon^{-1}\left(k,\omega\right)}{\omega} + \frac{1}{\pi z}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\epsilon^{-1}\left(k,\omega\right)}{\omega - z}d\omega \ ,$$

we have the canonical solution of the moment problem:

$$\epsilon^{-1}\left(k,z\right) = 1 + \frac{\omega_{pd}^{2}\left(z+Q_2\right)}{z\left(z^2-\omega_2^2\right)+Q_2\left(z^2-\omega_1^2\right)}. \tag{22}$$

- It is quite clear from Eq. (22) that the parameter function $Q_2(k, z)$ modifies both the position and the width of the lines in the spectrum of collective excitations of the system to be determined from the dispersion equation

$$z(z^2 - \omega_2^2) + Q_2(z^2 - \omega_1^2) = 0. \quad (23)$$

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- It is also important that (22) satisfies the sum rules independently of the choice of the parameter function $Q_2(k,z)$, such that

$$Q_2(k,z)=b(k)+ih(k)+\int_{-\infty}^{\infty}\frac{dg(t)}{t-z},\;\; b\in\mathbb{R}\;,\; h>0, \tag{24}$$

$$\lim_{z\to\infty}Q_2(k,z)/z=0,\;\int_{-\infty}^{\infty}\frac{dg(t)}{1+t^2}<\infty,\;\mathrm{Im}\,z>0. \tag{25}$$

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$$\lim_{z \to \infty} Q_2(k, z)/z = 0, \; \int_{-\infty}^{\infty} \frac{dg(t)}{1 + t^2} < \infty, \; \mathrm{Im}\, z > 0. \tag{25}$$

- If we neglect the processes of energy absorption completely, we should put $Q_2(k, z) = i0^+$. Mathematically this means the application, instead of the Nevanlinna formula (22), of the canonical solution.

- On the other hand, the corresponding hydrodynamic model leads [5], for the effective biexponential potential (12), to the following dispersion equation:

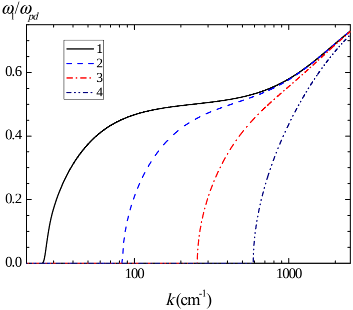
$$\omega\left(\omega + i\nu_d\right) = k^2\left[c^2 + \omega_{pd}^2\left(\frac{\theta_1}{k^2 + k_1^2} + \frac{\theta_2}{k^2 + k_2^2}\right)\right] \equiv \omega_{HD}^2\left(k\right), \tag{26}$$

where

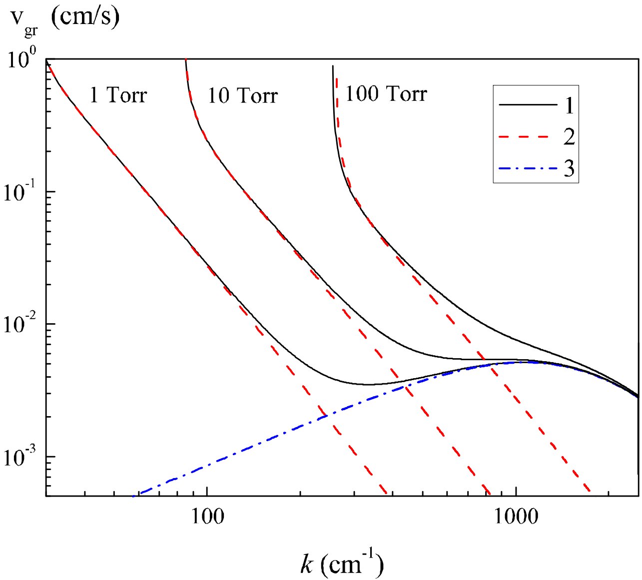
$$\nu_d = \frac{6\pi\eta_0 R_d}{m_d}, \qquad c^2 = \frac{1}{m_d}\left(\frac{\partial p}{\partial n_d}\right). \tag{27}$$

η_0 being the neutral gas dynamic viscosity and p - the pressure in the system.

The real part of the corresponding solution is displayed in the following figure for $T_e = T_i = 300\ K$, $R_d = 10\mu m$, $\nu_d = \omega_{pd} = 13.10\ s^{-1}$, $Z_d = -193.37$, $\theta_1 \approx \theta_2$, $k_1 = 2190.31\ cm^{-1}$, for different values of the smaller screening parameter: 1 – $p = 1\ Torr$, $k_2 = 26.61\ cm^{-1}$, 2 – $p = 10\ Torr$, $k_2 = 84.13\ cm^{-1}$, 3 – $p = 100\ Torr$, $k_2 = 266.07\ cm^{-1}$, 4 – $p = 760\ Torr$, $k_2 = 733.49\ cm^{-1}$.

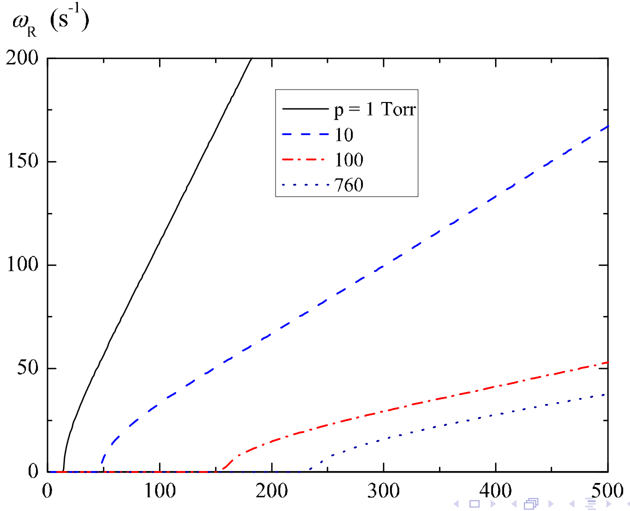


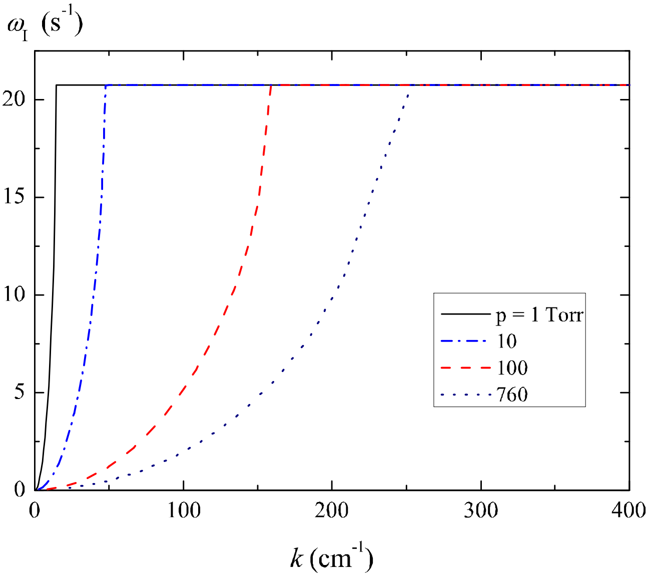
These results imply the possibility of observation of two different dust acoustic waves with different velocities and group velocities:



For large values of Γ we should take into account the non-ideality in the EOS and solve the "non-ideal" HD dispersion equation

$$\omega\left(\omega + i\nu_d\right) = k^2\left[\frac{1}{m_d}\left(\frac{\partial p}{\partial n_d}\right)_T + \omega_{pd}^2\left(\frac{\theta_1}{k^2+k_1^2} + \frac{\theta_2}{k^2+k_2^2}\right)\right] : \quad (28)$$



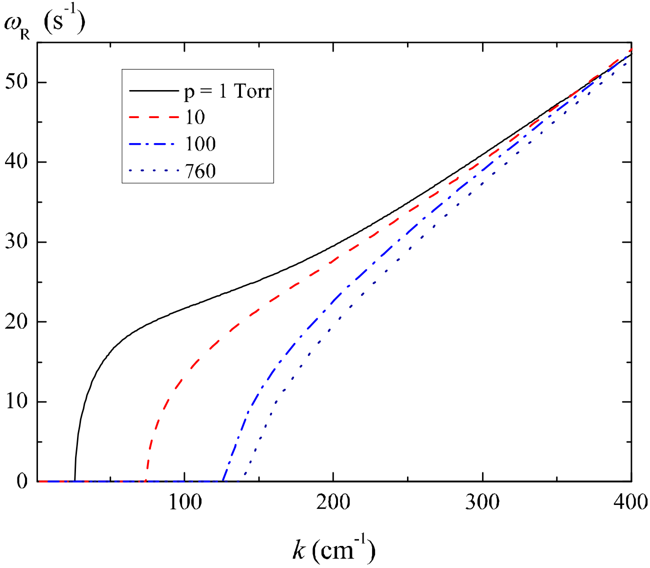


We see that the non-ideality does not remove the posibility of observation of two velocities for the DAWs.

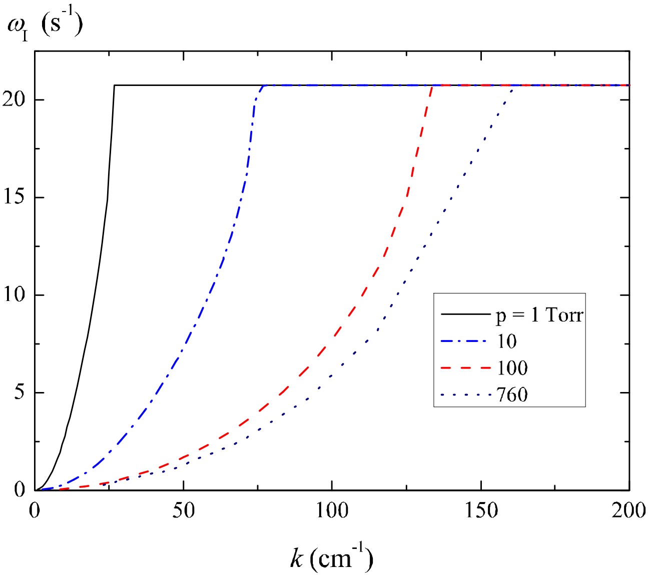
We have also studied the simplified dispersion equation of the method of moments

$$\omega\left(\omega + i\nu_d\right) = \omega_{pd}^2\left[\zeta\left(k\right) + V\left(k\right) + U\left(k\right)\right] =$$
$$= c_s^2 k^2 + \omega_{pd}^2\left[U\left(k\right) + k^2\left(\frac{\theta_1}{k^2 + k_1^2} + \frac{\theta_2}{k^2 + k_2^2}\right)\right], \qquad (29)$$

wherefrom we found the real,



and imaginary part of the solution of this simplified version of the dispersion equation stemming from the method of moments,



- We observe some numerical difference between the dispersion relations stemming from the simplified version of the method of moments and the hydrodynamic model.

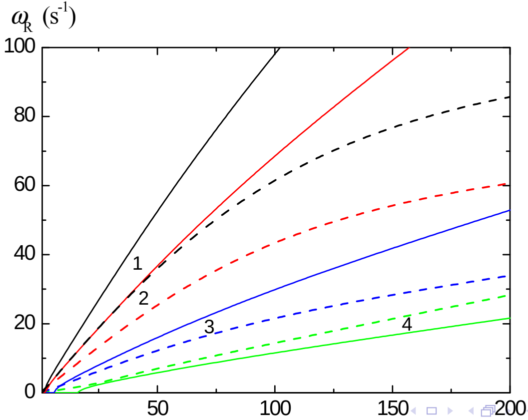
To resolve this discrepancy we approximated the Nevanlinna parameter function by its static value,

$$Q_2\left(k,0\right) = i\frac{\nu_d}{2}\frac{\omega_2^2\left(k\right)}{\omega_1^2\left(k\right)}$$

and solved the cubic dispersion equation (23)

$$z(z^2 - \omega_2^2) + Q_2(z^2 - \omega_1^2) = 0.$$

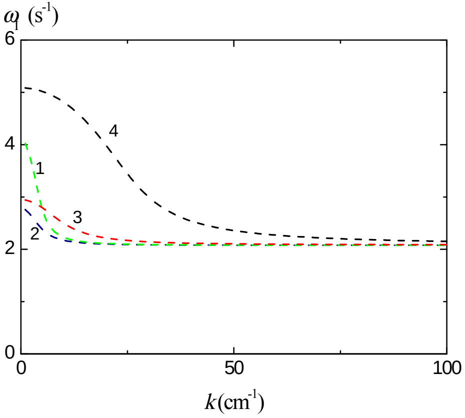
The results, at least for lower values of the ionization velocity, are encouraging. Precisely, for $1 - n_d = 10^6 cm^{-3}$, $1 - 5 \cdot 10^5 cm^{-3}$, $3 - 10^5 cm^{-3}$, $4 - 10^4 cm^{-3}$, with the hydrodynamic model result displayed as solid lines and the real part of the solution of the cubic equation shown as dashed lines ($Q_{ion} = 10^{12} cm^{-3} s^{-1}$, $p = 760\ Torr$, $\nu_d = 4.1 s^{-1}$) we obtained a reasonable agreement between these two approaches:



Additionally, the cubic equation

$$z(z^2 - \omega_2^2) + Q_2(z^2 - \omega_1^2) = 0.$$

generated a third, purely imaginary, root $i\omega_{\rm I}(k)$ which can be interpreted as the diffusive non-propagating mode (for the same conditions):



- Other models for the Nevanlinna parameter function are being studied currently.

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- Experimental verification of our results is also planned at the TRINITI.

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